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Two Exact Models of Charged Dust Collapse

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Liang studied three classes of irrotational dust collapse models with high symmetries. Here the charged analogs of the models with spherical and plane symmetry are considered. Contrary to Liang's result that the plane-symmetric model with positive mass cannot have a static exterior we find that the corresponding charged model may have bounce and a static exterior.

1. INTRODUCTION

Exterior solutions for the plane-symmetric Einstein equations were obtain by Taub (1951). Takeno (1960) and others studied the gravitational wave equations. Exterior solutions of Einstein–Maxwell equations exhibiting plane symmetry were developed by Patnaik (1970) and Letelier and Tabensky (1974) for static and nonstatic cases, respectively, while the interior solution was obtained by Humi and Le Britton (1975). Liang (1974) discussed the gravitational collapse of three cases of highly symmetric, but inhomogeneous models: spherical, plane, and cylindrical. The interior solutions were joined smoothly with the respective exterior ones. The plane-symmetric model was found to have either negative gravitating mass and bounce or had no static exterior. The plane-symmetric model may be of interest in the theory of galactic structure, which involves both matter and electromagnetic field. We therefore thought it worthwhile to study the evolution of the plane-symmetric model in the presence of an electromagnetic field. Our solution may be treated as the charged analog of Liang's solution. When one of the factors in our metric vanishes it reduces to Liang's charge-free solution. Liang had to make the unphysical assumption of negative mass in order to get a static exterior. But in the presence of electromagnetic field such an assumption is not necessary. The plane-symmetric solution is discussed in Section 2.

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In Section 3 the charged analog of the spherical model is discussed briefly for comparison. The external field is just the Reissner–Nordström solution (1916) and this case has already been studied in detail by various authors (Bardeen, 1968; Novikov, 1967). The charged dust distribution cannot collapse into the singularity $R=0$ because the repulsive electrostatic force ultimately overcomes the attractive force as R decreases. The sphere reaches a minimum volume inside the inner horizon and reexpands into another asymptotically flat universe.

2. THE PLANE-SYMMETRIC SOLUTION

In comoving normal coordinates the plane-symmetric line element may be taken as

$$ds^2 = dt^2 - e^{2\psi} dz^2 - \alpha^2(dx^2 + dy^2) \tag{2.1}$$

where $\psi = \psi(z,t)$, $\alpha = \alpha(z,t)$, and

$$(x^1, x^2, x^3, x^4) \to (x, y, z, t)$$

$$u^\mu = \delta_4^\mu \qquad \text{(dust flow velocity)}$$

The Einstein–Maxwell equations we discuss are

$$G_\nu^\mu = -8\pi(T_\nu^\mu + E_\nu^\mu) \tag{2.2}$$

$$T_\nu^\mu = \rho u^\mu u_\nu \tag{2.3}$$

$$E_\nu^\mu = \tfrac{1}{4}\delta_\nu^\mu F_{\alpha\beta} F^{\alpha\beta} - F^{\mu\alpha} F_{\nu\alpha} \tag{2.4}$$

$$F^{\mu\nu}_{\ ;\nu} = 4\pi J^\mu ; F_{[\mu\nu;\alpha]} = 0, \qquad J^\mu = \sigma(x^\alpha)\delta_4^\mu \tag{2.5}$$

where all symbols have the usual meaning (we have taken $G=c=1$).

If it is assumed that $F_{\mu\nu}$ and J^μ carry the same plane symmetry as the metric, then the only nonvanishing components of $F_{\mu\nu}$ are

$$F_{12} = c_1, \qquad F_{43} = \frac{c_2(z)e^\psi}{\alpha^2} \tag{2.6}$$

where c_1 is a constant and c_2 is a function of z alone.

The only nonvanishing components of $E_{\mu\nu}$ are

$$E_1^1 = E_2^2 = -E_3^3 = -E_4^4 = \tfrac{1}{2}\alpha^{-4}(c_1^2 + c_2^2) \tag{2.7}$$

We get corresponding to $G_{43}=0$

$$\frac{\partial \alpha}{\partial z}=0 \tag{2.8a}$$

and

$$e^{\psi}=\frac{\alpha'}{k(z)} \tag{2.8b}$$

where $k(z)$ is an arbitrary function of z. Taking the more general solution (2.8b) we get from the field equations

$$\frac{kk'}{\alpha\alpha'}-\frac{\ddot{\alpha}}{\alpha}-\frac{\dot{\alpha}\dot{\alpha}'}{\alpha\alpha'}-\frac{\ddot{\alpha}'}{\alpha'}=\frac{c}{\alpha^4} \tag{2.9}$$

$$\frac{1}{\alpha^2}(2\alpha\ddot{\alpha}+\dot{\alpha}^2-k^2)=\frac{c}{\alpha^4} \tag{2.10}$$

$$\frac{\dot{\alpha}^2}{\alpha^2}+\frac{2\dot{\alpha}\dot{\alpha}'}{\alpha\alpha'}-\frac{2kk'}{\alpha\alpha'}-\frac{k^2}{\alpha^2}=\frac{c}{\alpha^4}+8\pi\rho \tag{2.11}$$

where the prime denotes $\partial/\partial z$ and the dot denotes $\partial/\partial t$, and $c=4\pi(c_1^2+c_2^2)$ is a function of z alone. The field equations (2.9)–(2.11) are not independent; in fact after multiplying (2.10) by α^2 and then differentiating with respect to z we obtain an equation which if added to (2.9) multiplied by $2\alpha\alpha'$ leads to

$$\frac{\partial c}{\partial z}=0$$

Thus, $c=\text{const}$ irrespective of whether $\rho=0$ or $\rho\neq 0$. The equation (2.10) gives as a first integral

$$\tfrac{1}{2}\dot{\alpha}^2-\frac{m(z)}{\alpha}+\frac{\beta}{\alpha^2}=E(z) \tag{2.12}$$

where $m(z)$ is an integration function and

$$E(z)=(1/2)k^2(z)\geqslant 0, \qquad c/2=\beta$$

The singularity in equation (2.8b) is still avoided when both $k(z)$ and $\alpha'(z,t)$ vanish but the ratio exists. When $k(z)=\alpha'(z,t)=0$ for all z and $\beta=0$, we get the homogeneous flat Friedmann universe. From the source equation (2.11), we obtain

$$8\pi\rho = \frac{2m'(z)}{\alpha^2\alpha'}$$

whence

$$m(z) = 4\pi\int k\rho d^3v + m(z_0) \tag{2.13}$$

where $d^3v = dx\,dy\,dz\,e^{\psi}\alpha^2$ is the proper three volume. Here the lower limit in the integral (2.13) is taken as z_0 from where ρ starts to become nonzero because unlike the spherical case here there is no compelling reason to choose an origin. Further, we integrate over dx and dy over the range of volume 0 to 1. Similarly, the total charge Q may be given by

$$Q = c_2 = \int 4\pi\sigma d^3v + Q(z_0)$$

Let us suppose that $\rho = 0$ outside some exterior world tube of flow lines defined by $z = z_0$ in the electrovac exterior. We have from equation (2.13), $m = M$ (a constant) and $c = \text{const}$. Two coordinates are defined such that

$$\alpha = \alpha(z,t), \qquad T = T(z,t)$$

such that

$$\dot{T} = (2M/\alpha - c/\alpha^2)^{-1}(\dot{\alpha}^2 - 2M/\alpha + c/\alpha^2) \tag{2.14}$$

$$T' = \dot{\alpha}\alpha'(2M/\alpha - c/\alpha^2)^{-1}(\dot{\alpha}^2 - 2M/\alpha + c/\alpha^2)^{-1/2} \tag{2.15}$$

where the integrability condition of T is satisfied via the field equations. In the (T,α) coordinates the metric becomes

$$ds^2 = \frac{\alpha^2}{2M\alpha - c}\,d\alpha^2 - \alpha^2(dx^2 + dy^2) - \frac{2M\alpha - c}{\alpha^2}\,dT^2 \tag{2.16}$$

This metric is in fact a special case of charged analog of the Kasner universe (Kasner, 1925).

When $c = 0$ we get the charge-free solution found by Liang. In this case α is timelike for all positive values of M and hence the metric is time dependent. So in order to have a static exterior solution the mass M must be negative. This assumption is unphysical. However, since $c > 0$, α is timelike when $2M\alpha > c$ and spacelike when $2M\alpha < c$. Hence we cannot have a globally static exterior solution. It should be noted that the hypersurface $\alpha = c/2M$ is null and plays the role of a Killing horizon. For $\alpha > c/2M$ the Killing vector is spacelike, while for $\alpha < c/2M$ it is timelike. However, for the unchanged case, i.e., when $c = 0$, the Killing vector is always spacelike for $M > 0$.

3. SPHERICAL SYMMETRY

Using comoving normal coordinates the spherically symmetric line element may be written in the form

$$ds^2 = dt^2 - e^{2\psi} dr^2 - R^2 d\Omega^2 \tag{3.1}$$

where $\psi = \psi(r,t)$, $R = R(r,t)$. We get from the field equations

$$e^{\psi} = R'/k(r) \tag{3.2}$$

$$\frac{1}{2}\dot{R}^2 - \frac{m(r)}{R} + \frac{c(r)}{2R^2} = E(r) \tag{3.3}$$

where $E(r) = (1/2)(k^2 - 1) \geqslant -1/2$ and c is a function of r alone, connected with electromagnetic field. From the source equation we obtain

$$\rho = \frac{m'(r)}{4\pi R^2 R'} \tag{3.4}$$

where the symbols have their previous meaning.

The equation (3.4) gives

$$m(r) = \int_0^r \rho k d^3 v \tag{3.5}$$

where $d^3 v = 4\pi R^2 e^{\psi} dr$.

For the exterior we get from equation (3.4) $m = M$, (a constant) and $c = \text{const}$.

We then define new coordinates

$$R = R(r,t), \qquad T = T(r,t)$$

such that

$$\dot{T}=\frac{\left(1+\dot{R}^{2}-2M/R+2c/R^{2}\right)^{1/2}}{1-2M/R+2c/R^{2}} \tag{3.6a}$$

$$T'=\frac{\dot{R}R'}{\left(1-2M/R+2c/R^{2}\right)\left(1+\dot{R}^{2}-2M/R+2c/R^{2}\right)} \tag{3.6b}$$

where the integrability condition is again ensured via the field equations.

In the new coordinates the metric reduces to

$$ds^{2}=\left(1-2M/R+2c/R^{2}\right)dT^{2}-\left(1-2M/R+2c/R^{2}\right)^{-1}dR^{2}-R^{2}d\Omega^{2} \tag{3.7}$$

which is the well-known Reissner–Nordström solution.

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REFERENCES

Bardeen, J. M. (1968). *Bulletin of the American Physical Society*, **13**, 41.
Humi, M., and Le Britton, L. (1975). *Physical Review*, **110** 2689.
Kasner, E. (1925). *Transactions of the American Mathematical Society*, **27** 155.
Liang, E. P. T. (1974). *Physical Review*, **100**, 447.
Letelier, P., and Tabensky, R. (1974). *Journal of Mathematical Physics*, **15**, 594.
Novikov, I. (1967). *Soviet Astronomy-AJ*, **10**, 731.
Patnaik, S. (1970). *Proceedings of the Cambridge Philosophical Society*, **67**, 127.
Reissner, H. (1916). *Annalen der Physik* (*Leipzig*), **50**, 106.
Taub, A. H. (1951). *Annals of Mathematics*, **53**, 472.
Takeno, H. (1960). *Tensor*, **10**, 34.